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A REMARK ON THE BEHAVIOUR OF L^p-MULTIPLIERS AND THE RANGE OF OPERATORS ACTING ON L^p-SPACES

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ABSTRACT

In this paper, new results are obtained concerning the uniform approximation property (UAP) in L^p-spaces ($p \neq 2, 1, \infty$). First, it is shown that the "uniform approximation function" does not allow a polynomial estimate. This fact is rather surprising since it disproves the analogy between UAP-features and the presence of "large" euclidian subspaces in the space and its dual. The examples are translation invariant spaces on the Cantor group and this extra structure permits one to replace the problem with statements about the nonexistence of certain multipliers in harmonic analysis. Secondly, it is proved that the UAP-function has an exponential upper estimate (this was known for $p = 1, \infty$). The argument uses Schauder's fix point theorem. Its precise behaviour is left unclarified here. It appears as a difficult question, even in the translation invariant context.

1. Introduction

The aim of this note is to settle some problems which came up in the paper [BT] and especially in the preceding paper [GTT] to which we also refer for background. We will show the following fact.

THEOREM: *Let $1 < p < \infty$, $p \neq 2$. For all $\delta > 0$ and N sufficiently large, there is a subspace X of the N-dimensional L^p-space ℓ^p_N, such that* codim $X < N^\delta$ *and any well-bounded operator T on ℓ^p_N ranging in X satisfies* trace $T = o(N)$.

We omit a heavy explicit formulation of the previous statement since it is clear to the reader a bit familiar with this area. Otherwise he may consult [GTT].

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In fact, the proof yields quantitative information that would permit to state a stronger result.

Let us list some consequences of such examples.

(1) Let $1 < p < 2$ and take $\delta = \frac{2}{p'}$. It is known that if X has codimension $< N^{\frac{2}{p'}}$, its Euclidean distance is extremal, i.e.

$$\text{dist}\,(X, \ell^2{}_{\dim X}) \sim N^{\frac{1}{p}-\frac{1}{2}}. \tag{1.1}$$

On the other hand, X does not contain an isomorph of ℓ^p_n for $n \sim N$. Indeed, since this would imply a complemented embedding of an ℓ^p_n-subspace (complemented in the initial ℓ^p_N-space), see [BT], there would be an X-valued operator on ℓ^p_N with trace $T \sim N$. This question was left open in [BT] and was raised in [JS].

(2) Denote $X^\perp$ the annilator of X given by the theorem. Thus $Y = X^\perp$ is an $n \equiv$ codim$X$ dimensional subspace of $\ell^{p'}_N$. Assume S is an operator on $\ell^{p'}_N$ satisfying

$$S\,|_Y = \text{ identity}. \tag{1.2}$$

Clearly the operator on ℓ^p_N given by

$$T = \text{ Id } - S^* \tag{1.3}$$

has range in X. Thus

$$\text{trace}T = o(N)$$

and therefore in particular

$$\text{trace}S \sim N. \tag{1.4}$$

Assuming $\delta > 0$ given and N large enough, one gets thus a subspace Y of $\ell^{p'}_N$, $\dim Y = n < N^\delta$, such that if S fulfills (1.2), then (1.4) holds, i.e.

$$\text{trace}S > n^{1/\delta}. \tag{1.5}$$

This fact means that for $p \neq 2$ the space L^p does not have the polynomial approximation property (p' plays the role of p in the considerations above). This question was open, except for $p = 1, \infty$ in which case it is known (see [FJS]) that for certain n-dimensional spaces Y (1.2) implies trace$S > e^{cn}$. In fact, the spaces L^1, L^∞ have an exponential UAP function. In Appendix, we show that the UAP function is exponentially bounded for all $L^p, 1 \leq p \leq \infty$. The question whether one really has an exponential behaviour is left open. It is not implied by the example described below.

(3) The statement of the theorem may be made more precise. One will show below (see 3.43) an estimate

$$\text{trace}T < N^{1-\gamma} \cdot \|T\| \qquad \text{where } \gamma = \gamma(\sigma) > 0. \tag{1.6}$$

As a corollary, one concludes that any Banach space possessing a "polynomial approximation property" needs to have type $2-\varepsilon$, cotype $2+\varepsilon$, for all $\varepsilon > 0$. Indeed, one would get otherwise by a result of Pisier*, for some $p \neq 2$, uniform l^p_N-isomorphs for $N \to \infty$ admitting a complementation at most $C_\tau N^\tau$, for any given $\tau > 0$. Hence the previous reasoning together with (1.6) easily yield a contradiction.

This observation was communicated to the author by W.B. Johnson [J1]. It is of course a natural question what may be said more, especially in view of the results of [JP] where the uniform approximation property with linear uniformity function is characterized.

The spaces appearing in the theorem are translation invariant spaces on the Cantor group. We will make essential use of harmonic Analysis methods and eventually will have to study the behaviour of certain L^p-multipliers. The results obtained there are of independent interest.

Throughout this note, letters c, C will denote numerical constants. If they may depend on p, we add p as subscript.

2. Description of the examples

We denote G the finite Cantor group $\{1,-1\}^n$ with $N = 2^n$ elements. The measure on G is the normalized Haar-measure (= product measure) and the characters are given by the Walsh functions

* This result (unpublished) is based on type–cotype theory. If the given space is B-convex, one may invoke a result from [P]. Otherwise, the polynomial AP hypothesis and the logarithmic bounds on K-convexity constants of finite dimensional spaces permit one to construct l^1_N-isomorphs in the space, which have a complementation constant at most $\log N$.

$$w_S(\varepsilon) = \prod_{i \in S} \varepsilon_i$$

where ε_i $(1 \leq i \leq n)$ is the ith coordinate projection and S ranges over the 2^n subsets of $\{1, \dots, n\}$. We identify ℓ^p_N with the space $L^p(G)$. For $\Lambda \subset 2^{(n)}$, the subspace L^p_Λ is generated by $\{w_S \mid S \in \Lambda\}$. Let $X = L^p_\Lambda$ where

$$\Lambda = \{S \subset \{1, \dots, n\} \mid |S| < n - m\}$$

($|S|$ stands for the cardinality of the set S). Here m is chosen to satisfy

$$\frac{m}{n} \log \frac{n}{m} \sim \delta \tag{2.1}$$

so that indeed

$$\operatorname{codim} X = \binom{n}{n-m} + \binom{n}{n-m+1} + \dots + \binom{n}{n} \sim \left(\frac{n}{m}\right)^m < N^\delta. \tag{2.2}$$

Suppose $T : L^p(G) \to X$ a linear operator such that

$$\operatorname{trace} T > \tau N. \tag{2.3}$$

Our first aim is to replace T by a multiplier. This is achieved the usual way by averaging, i.e. define

$$T_1 = \int_G (R_\varepsilon T R_\varepsilon) d\varepsilon$$

where R_ε denotes the translation operator

$$R_\varepsilon f(x) = f(\varepsilon \cdot x). \tag{2.4}$$

Thus T_1 corresponds to a multiplier $\lambda = (\lambda_S)$ where

$$\lambda_S = 0 \qquad \text{if } |S| \geq n - m, \tag{2.5}$$

$$\|\lambda\| \equiv \|\lambda\|_{p \to p} \leq \|T\|, \tag{2.6}$$

$$\sum \lambda_S = \operatorname{trace} T. \tag{2.7}$$

An additional averaging argument over the symmetric group Sym(n) by permuting the coordinates permits one to assume, moreover, that

$$\lambda_S = \lambda_{|S|}. \tag{2.8}$$

From (2.3), (2.7) it now follows that for some

$$\frac{1}{4}\,n < \overline{n} < \frac{3}{4}\,n \tag{2.9}$$

one has

$$\lambda_{\overline{n}} > \frac{\tau}{2} \tag{2.10}$$

assuming the condition

$$\log\,\frac{4\|T\|}{\tau} < \frac{n}{100} \tag{2.11}$$

satisfied.

Our aim is to show that for given $\tau, \delta > 0$ it is impossible for a multiplier λ on $L^p(G)$ to satisfy the conditions

$$\begin{cases} \|\lambda\| \leq 1 \quad (2.12), \\ \lambda_t = 0 \text{ if } t \geq n - m \quad (2.13), \\ (2.9) + (2.10). \end{cases}$$

Here m satisfies (2.1) and the normalization (2.12) may always be assumed. This is, of course, a concrete harmonic analysis problem on the Cantor group. We settle this question in the next section.

3. Estimates on certain multipliers

We will need two elementary lemmas. We don't attempt to state them in the most precise form.

LEMMA 3.1: *For $p \neq 2$, there is an integer $r = r_p > 10$ and a number $\beta = \beta_p > 1$ as well as a function $\varphi \in [w_S \mid S \subset \{1, \dots, r\}]$, such that*

$$\int \varphi = 0, \tag{3.2}$$

$$\|\varphi\|_p \leq 1, \tag{3.3}$$

$$\Big\| \sum_{i=1}^{r} \widehat{\varphi}(\{i\})\varepsilon_i \Big\|_p = \beta > 1. \tag{3.4}$$

Proof: The first condition (3.2) may be ignored, since one may always replace φ by $\frac{1}{2}\Big(\varphi(\varepsilon) - \varphi(-\varepsilon)\Big)$. Then (3.2) is achieved, $\|\varphi\|_p$ is non-increasing and (3.4) preserved. In this form, it suffices to observe that $L^p_{\{\{i\};1\leq i\leq r\}}$ is not norm-1 complemented in $L^p\big(\{1,-1\}^r\big)$ since this subspace is not spanned by functions having mutually disjoint support; see e.g. [L], Chap. 6.* ∎

LEMMA 3.5: *Let $a < b$ be positive integers and $M_{a,b}$ the smallest constant satisfying the polynomial inequality*

$$\sup_{-1<\delta<1} \Big|\sum_{j=a}^{b} p_j t^j\Big| > \frac{1}{M_{a,b}} \max |\, p_j \,| \tag{3.6}$$

for any polynomial written as a linear combination of the monomials $\{t^j \mid a \leq j \leq b\}$. Then one has an estimate

$$\log\ M_{a,b} < c\cdot (b-a)\log\Big(\frac{b}{b-a} + 2\Big).^{**} \tag{3.7}$$

This estimate (in fact the L^2-estimate, which is equivalent here) may be found in [N] (in the discussion of Müntz' theorem). Alternatively, one may proceed by Lagrange interpolation on the interval $[1 - \frac{b-a}{b}, 1]$.

Define now

$$\overline{m} = \Big[\frac{m}{r}\Big] \tag{3.8}$$

([ ] denotes the integer part).

Let $m_1 < \frac{1}{2}\,\overline{m}$ be a positive integer on which a more precise upper estimate will be imposed later. We assume

$$m < \frac{4}{5}\, n \tag{3.9}$$

* In fact, the conditions of the lemma may already be realized for $r = 3, \varphi \in [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1\varepsilon_2\varepsilon_3]$.

** The interest in this estimate is that it improves on the exponential estimate on b when $b - a = o(b)$. This is the situation in which it will be applied later on.

(in fact m is, of course, initially assumed small w.r.t. n but will be adjusted later on through some iteration process that finally will contradict (2.10)).

By (3.9), the integer n_1 defined by

$$n - m - m_1 = n_1 + \overline{m} \tag{3.10}$$

is clearly positive. Also, from (3.8), (3.10)

$$n_1 + r\,\overline{m} < n. \tag{3.11}$$

Define the function $(-1 \leq \delta \leq 1)$

$$\varphi_\delta = \sum_{S \subset \{1,\ldots,r\}} \delta^{|S|} \widehat{\varphi}(S) w_S = \varphi * (\Sigma \delta^{|S|} w_S) \tag{3.12}$$

where φ is the function given by Lemma 3.1. Of course φ_δ still satisfies (3.2), (3.3).

Taking (3.11) into account, consider the function f_δ on G defined as follows:

$$f_\delta(\varepsilon) = \tag{3.13}$$

$$\varphi_\delta(\varepsilon_1, \ldots, \varepsilon_r)\varphi_\delta(\varepsilon_{r+1}, \ldots, \varepsilon_{2r}) \cdots \varphi_\delta(\varepsilon_{(\overline{m}-1)r+1}, \ldots, \varepsilon_{\overline{m}r}) \cdot \varepsilon_{\overline{m}r+1} \cdots \varepsilon_{\overline{m}\cdot r+n_1}.$$

Thus

$$\|f_\delta\|_p \leq 1 \qquad \text{for } -1 \leq \delta \leq 1. \tag{3.14}$$

Define F_t by the equation

$$\delta^{t-n_1} F_t = \sum_{|S|=t} \widehat{f}(S) w_S. \tag{3.15}$$

It follows from (3.2), (3.10) that

$$F_t = 0 \qquad \text{if } t < n - m - m_1 \tag{3.16}$$

and

$$F_{n-m-m_1} = F_{\overline{m}+n_1} = \left[\sum_1^r \widehat{\varphi}(i)\ \varepsilon_i\right] \cdot \left[\sum_1^r \widehat{\varphi}(i)\ \varepsilon_{r+i}\right] \cdots \varepsilon_{\bar{m}r+1} \cdots \varepsilon_{\bar{m}\cdot r+n_1}. \tag{3.17}$$

From (3.4), clearly by the probabilistic independence of the factors in (3.17)

$$\|F_{n-m-m_1}\|_p = \beta^{\bar{m}}. \tag{3.18}$$

Consider the convolution

$$f_\delta * \lambda = \sum_{n-m-m_1 \le t < n-m} \delta^{t-n_1} \lambda_t F_t. \tag{3.19}$$

In this summation, the lower restriction on t follows from (3.16) and the upper from (2.13).

Again from (3.10), (3.19) is a (vector-valued) polynomial in $\delta \in [-1,1]$ written in monomials δ^j with $a = \bar{m} \le j < \bar{m} + m_1 = b$. Since Lemma 3.5 carries obviously from the scalar to the vector valued setting (by polarization), the inequality

$$\|f_\delta * \lambda\|_p \le \|\lambda\| \tag{3.20}$$

implied by (3.14) yields, in particular,

$$|\lambda_{n-m-m_1}| \|F_{n-m-m_1}\|_p \le M_{\bar{m},\bar{m}+m_1} \|\lambda\|. \tag{3.21}$$

Thus, from (3.18), (3.7), (3.8)

$$|\lambda_{n-m-m_1}| < \beta^{-\bar{m}} \cdot \left(\frac{\bar{m}}{m_1}\right)^{c \cdot m_1} \|\lambda\| < \beta^{-\frac{\bar{m}}{2}} \|\lambda\| < 2^{-c_p m} \|\lambda\| \tag{3.22}$$

provided m_1 is subject to a restriction of the form $m_1 < c_p \bar{m}$, or equivalently

$$m_1 < c_p\, m. \tag{3.23}$$

It follows from (2.12), (3.22) that for m_1 satisfying (3.23)

$$|\lambda_{n-m-m_1}| < 2^{-c_p m}. \tag{3.24}$$

Fix a positive integer ℓ (to be specified) and introduce a new multiplier μ defined by

$$\begin{cases} \mu_S = \lambda_S^\ell & \text{if } |S| \le n - m - c_p m, \\ \mu_S = 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

From (3.24) and harmonic analysis on the Cantor group, it follows that for $p^* = p \vee p'$

$$(3.27) \qquad \|\mu\| \le \|\lambda\|^\ell + \sum_{n-m-c_p m<r<n-m} (p^*-1)^{\frac{n-r}{2}} |\lambda_r|^\ell < 1 + 2^{-c_p \ell m} c_p^m$$

and for an appropriate choice

$$(3.28) \qquad \ell = \ell_p$$

one gets

$$(3.29) \qquad \|\mu\| < 2.$$

Considering $\frac{1}{2}\mu$, a new multiplier is obtained, satisfying (2.12), (2.13) with m replaced by $(1+c_p)m$. It follows from (2.10), (3.24) and the definition of μ that

$$(3.30) \qquad \bar{n} < n - m - c_p m; \qquad \mu_{\bar{n}} = \lambda_{\bar{n}}^\ell > \left(\frac{\tau}{2}\right)^\ell.$$

Thus $\frac{1}{2}\mu$ fulfills (2.10) with τ replaced by

$$(3.31) \qquad \tau_1 = \left(\frac{\tau}{2}\right)^\ell.$$

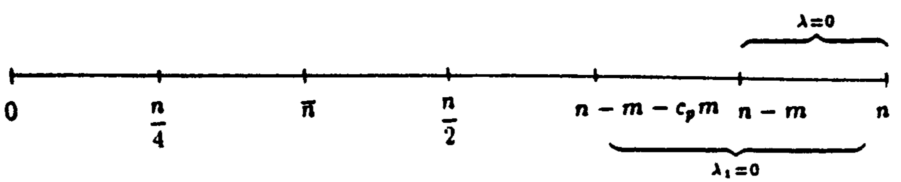
In what precedes, we used our assumption

$$(3.32) \qquad 2^{-c_p m} < \frac{\tau}{2},$$

i.e., from (2.1)

$$(3.33) \qquad N > \left(\frac{1}{\tau}\right)^{c_p \cdot \frac{1}{\delta} \cdot \log \frac{1}{\delta}}$$

Repeat all previous considerations with λ replaced by $\lambda_1 = \frac{1}{2}\mu$.



This is possible provided the analogue of (3.32) holds, thus by (3.31)

$$(3.34) \qquad 2^{-c_p(1+c_p)m} < \frac{\tau_1}{2} = \frac{1}{2}\left(\frac{\tau}{2}\right)^\ell.$$

Moreover, one has now

$$\bar{n} < n - m - c_p m - c_p(1 + c_p)m. \tag{3.35}$$

Iterate this construction d times. This will require to have an estimate of the form

$$2^{-c_p(1+c_p)^d m} < \left(\frac{\tau}{2^d}\right)^{\ell^d} \tag{3.36}$$

valid, which in turn is implied by a bound of the form

$$\frac{1}{\delta} \cdot \log \frac{1}{\delta} \cdot C_p^d \cdot \log \frac{1}{\tau} < n. \tag{3.37}$$

We also assume d satisfying

$$(1 + c_p)^d m < \frac{4}{5} n \tag{3.38}$$

in order to preserve condition (3.9).

For such d, one gets the analogue of (3.35)

$$\frac{n}{4} < \bar{n} < n - m - c_p m - c_p(1 + c_p)m - \cdots - c_p(1 + c_p)^{d-1}\, m = n - (1 + c_p)^d m \tag{3.39}$$

hence

$$(1 + c_p)^d \cdot m < \frac{3}{4} n. \tag{3.40}$$

Assuming $c_p > 0$ taken sufficiently small, (3.38) may be fulfilled and (3.40) not, unless (3.37) forces d to satisfy

$$(1 + c_p)^d < \frac{10}{\delta} \log \frac{1}{\delta}. \tag{3.41}$$

This is clearly not the case if n is assumed large enough. Thus one gets (3.37), (3.38) and the failure of (3.40), a contradiction. This completes the argument.

Observe that the preceding yields a more precise condition, of the form

$$n > \log \frac{1}{\tau} \cdot \left(\frac{1}{\delta}\right)^{c_p}, \tag{3.42}$$

to get the contradiction. Equivalently, there is a subspace X of ℓ_N^p of codimension $< N^\delta$ such that any operator T on ℓ_N^p ranging into X satisfies a trace estimate

$$\text{trace} T < N^{1-\gamma_p(\delta)} \|T\| \tag{3.43}$$

where

$$\gamma_p(\delta) > \delta^{C_p}. \tag{3.44}$$

4. Remark

We would like to draw the reader's attention to the following questions on $L^p(\{1,-1\}^n)$-multipliers, $p \neq 2, 1, \infty$.

Q_1. Is there $\varepsilon = \varepsilon_p > 0$ such that if λ is a $1+\varepsilon$-bounded multiplier on L^p and

$$\lambda_{\{i\}} = 1 \qquad \text{for } i = 1, \dots, n \tag{4.1}$$

then log trace $\lambda \sim n$?

Q_2. Is there a constant $K = K_p$ such that if λ is a 2-bounded multiplier on L^p and

$$\lambda_S = 1 \qquad \text{if } |S| \leq K_p \tag{4.2}$$

then log trace $\lambda \sim n$?

These were communicated to the author by W.B. Johnson. A positive answer to Q_2 solves Q_1 affirmatively.

5. Appendix: Exponential estimate on the UAP-function in L^p

It follows from the preceding that the UAP-function in L^p, say $\varphi_p(n)$, satisfies $\varphi_p(n) > \exp(\log n)^{1+c_p}$ ($p \neq 2, c_p > 0$), for $n \to \infty$. Our purpose is to show an exponential upper estimate (known for $p = 1, \infty$). Besides more "standard techniques" the argument will rely on a Schauder fixpoint argument.

Fix $2 < p < \infty$ (the case $1 < p < 2$ is covered by Mascioni's duality result, [M]). Let Ω be finite with normalized counting measure λ and X an n-dim function space on Ω. Let $K = \{\rho : \Omega \to [\frac{1}{2}, \infty[\mid \int \rho d\lambda = 1\}$, clearly compact, convex. For given $\rho \in K$, consider the set of triplets $\mathcal{K}_\rho = \{(E, T, \rho_1)\}$ where

(i) E is an operator with for all q

$$\|E\|_{L^q(\rho d\lambda) \to L^q(\rho d\lambda)} \leq 2,$$

$$\|E\|_{N\left(L^q(\rho d\lambda), L^q(\rho, d\lambda)\right)} \leq M.$$

(ii) T is an operator with

$$\|T\|_{L^\infty(\Omega) \to L^\infty(\Omega)} \leq \tfrac{1}{10}.$$

(iii) $I - E|_{\rho^{-1/p}X} = T|_{\rho^{-1/p}X}$.

(iv) $\rho_1 \in K$.

(v) $\int |Tf|^2 \rho d\lambda \leq \frac{1}{10} \int |f|^2 \rho_1 d\lambda \quad \forall f$

We make the following claims:

(I) $\mathcal{K}_\rho$ is convex and compact.

(II) For suitable choice of M (exponential in n), $\mathcal{K}_\rho$ is non-void and depends continuously on ρ.

Define then $K_\rho \subset K$ as the projection of $\mathcal{K}_\rho$ on the ρ_1 factor.

Hence K_ρ is non-void compact convex with continuous ρ-dependence.

One then defines a transformation of K, letting

$$\tau(\rho) = \text{ element of } K_\rho \text{ with minimum } L^2(\lambda) - \text{norm.}$$

From the preceding, $\tau(\rho)$ is continuous in ρ and hence there is a fix point, i.e., there is $\rho \in K$ with $\rho \in K_\rho$.

From (v), $\|T\|_{L^2(\rho d\lambda) \to L^2(\rho d\lambda)} < \frac{1}{\sqrt{10}}$ and interpolating with (ii) yields

$$\text{(vi)} \qquad \|T\|_{L^p(\rho d\lambda) \to L^p(\rho d\lambda)} < \frac{1}{\sqrt{10}} \qquad (2 \leq p \leq \infty).$$

The operator

$$E_1 = (I + T + T^2 + \cdots)E$$

is well bounded on $L^p(\rho d\lambda)$ with nuclear norm estimate M*, because of (i). Also $E_1|_{\rho^{-1/p}X} = \text{Id}\,|_{\rho^{-1/p}X}$ because of (iii). Thus the isometry $L^p(\lambda) \to L^p(\rho\lambda)$: $f \to \rho^{-1/p} f$ maps X to a space with desired approximation properties; therefore X itself satisfies them. It remains to show (II).

Assume $\rho \in K$, $(E, T, \rho_1) \in \mathcal{K}_\rho$ and $\rho' \in K$ an approximation of ρ such that

$$\text{(vii)} \qquad 1 - \epsilon < \frac{\rho}{\rho'} < 1 + \epsilon$$

(elements of K are $\geq \frac{1}{2}$). Denote $X_\rho = \rho^{-1/p} X, X_{\rho'} = (\rho')^{-1/p} X$,

$$E' = \left(\frac{\rho}{\rho'}\right)^{1/p} \cdot E\left(\left(\frac{\rho'}{\rho}\right)^{1/p}\right), \qquad T' = \left(\frac{\rho}{\rho'}\right)^{1/p} T\left(\left(\frac{\rho'}{\rho}\right)^{1/p}\right).$$

It easily follows from (vii) that E', T' and ρ_1 satisfy (i), (ii), (v) (replacing ρ by ρ', provided the estimates are multiplied by a factor $1 + C\epsilon$. Also, by construction $I - E'|_{X_{\rho'}} = T'|_{X_{\rho'}}$. Obviously (E', T', ρ_1) is a perturbation of (E, T, ρ_1). Assume

* Passing from the exponential nuclear norm estimate to exponential rank is achieved following [M].

we also have $(E'', T'', \rho'') \in \mathcal{K}_{\rho'}$ where moreover the estimates in (i), (ii), (v) are respectively

$$\text{(viii)} \qquad 1, \frac{1}{2}M, \frac{1}{20}, \frac{1}{20}.$$

Define then

$$\bar{E} = (1-\delta)E' + \delta E'',$$

$$\bar{T} = (1-\delta)T' + \delta T'',$$

$$\bar{\rho} = \frac{(1-\delta)(1+C\varepsilon)\rho_1 + \delta\frac{1}{2}\rho''}{(1-\delta)(1+C\varepsilon)+\frac{1}{2}\delta}.$$

For small δ, $(\bar{E}, \bar{T}, \bar{\rho})$ is a perturbation of (E', T', ρ_1), hence of (E, T, ρ_1). Here $\delta \sim \varepsilon$. Hence if we show that $(\bar{E}, \bar{T}, \bar{\rho}) \in \mathcal{K}_{\rho'}$, the continuous dependence of $\mathcal{K}_\rho$ on ρ is shown. Now (iii), (iv) are preserved under convex combination. The other properties follow from an estimate

$$(1-\delta)(1+C\varepsilon) + \frac{\delta}{2} < 1 \quad (\text{valid for some } \delta \sim \varepsilon).$$

The existence of (E'', T'', ρ'') is shown in a direct way. Take an expectation E'' w.r.t. $\rho' d\lambda$ of rank $\frac{M}{2}$ (exponential in $n = \dim X_{\rho'}$) such that

$$\|x - E''x\|_\infty \leq 10^{-4}\|x\|_\infty \qquad \text{for } x \in X_{\rho'}.$$

Its existence follows from entropy considerations in the unit ball of the dual space $X^*_{\rho'}$. Extend $I - E''|_{X_{\rho'}}$ to an operator T'' on $L^\infty(\Omega)$ with $|T''| \leq 10^{-4}$ and apply Grothensheck's theorem to get ρ''' s.t. $\rho''' \geq 0$, $\int \rho''' d\lambda = 1$ and

$$\int |T''f|^2 \rho' d\lambda \leq K_G^2 10^{-4} \int |f|^2 \rho''' d\lambda.$$

Finally, put $\rho'' = \frac{1}{2}(1+\rho''')$. All the conditions are clearly fulfilled.

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